

# Structural Routability of $n$ -Pairs Information Networks

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## Abstract

Information does not generally behave like a flow in communication networks with multiple sources and sinks. However, it is often conceptually and practically useful to be able to associate separate data streams with each source-sink pair, with only routing and no coding performed at the network nodes. This raises the question of whether there is a nontrivial class of network topologies for which achievability is always equivalent to “routability”, for any combination of source signals and positive channel capacities. This paper considers possibly cyclic, directed, errorless networks with  $n$  source-sink pairs, mutually independent source signals, and a relaxed communication objective in terms of demanded information rates at sinks. The concept of *triangularizability* is introduced and it is shown that, if the network topology is triangularizable, then a given combination of source signals, demand rates and channel capacities is achievable if and only if the digraph supports a feasible multicommodity flow.

## I. INTRODUCTION

In an  $n$ -pairs communication network,  $n$  source signals must be conveyed to their corresponding sinks without exceeding any channel capacities. Until quite recently, the belief was that this was possible iff there existed a *routing* solution, i.e. if every bit generated by a source could be carried without modification, over channels and through network nodes, until it reached the sink. At a macroscopic level, this is equivalent to presuming the existence of a feasible *multicommodity flow* [1].

However, in [2], [3], an example was constructed of a multi-source, multi-sink communication network that did not admit a routing solution, but became admissible if nodes could perform modulo-2 arithmetic on incoming bits. This counter-intuitive result started the field of *network coding*, in which nodes are permitted to not just route incoming bits, but also to perform functions on them so as to better exploit the network structure and the available channel capacities.

It is now known that the achievable capacity regions of networks with multiple sources and sinks are not generally given by feasible multicommodity flows, although in [4], the capacity region of a special class of 3-layered, acyclic  $n$ -pairs networks was demonstrated to be given by multicommodity flows. In [5],  $n$ -pair networks were constructed with coding capacity much larger than the routing capacity. Other related work includes [6], in which a necessary and sufficient condition for broadcasting correlated sources over erroneous channels was found, and [7], in which linear network coding was shown to achieve capacity for a multicast network.

Notwithstanding the power of network codes, routing/multicommodity flow solutions are appealing in several respects. Most obviously they are simpler, because network nodes are not required to perform extra mathematical operations on arriving bits. In addition, because different data streams are not “hashed” together by means of some function, there is arguably less potential for cross-talk between different source-sink pairs, arising for instance from nonidealities during implementation in the physical layer. Furthermore,

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being able to treat information as a conservative fluid-flow could potentially provide a simple basis to analyze communication requirements in areas outside traditional multiterminal information theory, e.g. networked feedback control and multi-agent coordination/consensus problems - see, e.g. [8], [9].

These considerations raise the natural questions of whether there is a general class of network topologies on which achievability is always equivalent to the existence of a feasible multicommodity flow, and of how to characterise this class. This paper derives a partial answer to this question for possibly cyclic, directed, errorless networks with  $n$  source-sink pairs and mutually independent source signals. The communication objective is relaxed so that, instead of reconstructing the source signals with negligible probability of error at the sinks, the aim is for the marginal information rate supplied to each sink to exceed a specified positive demand.

The structural concept of *triangularizability* is introduced, and the main result (Thm. 1) is that if the network topology is triangularizable then the network is *structurally routable*, i.e. a given combination of source signals, demand rates and channel capacities is achievable iff the network supports a feasible multicommodity flow. This property inheres solely in the topology of the network and suits situations where channels, switches, transceivers and interfaces are expensive to set up and difficult to move, or where channel capacities, output demands and source-signal statistics are unknown.

The primary significance of this result is structural, not computational. Algorithms can be devised to check whether or not a given network topology is triangularizable, by working directly from the definition. However, the explicit construction of such procedures and the analysis of their computational complexity will be addressed elsewhere.

Note that although triangularizability is sufficient to guarantee that routing can always achieve the full capacity of a network, it is not necessary, and the question of finding a structural condition that is less conservative - or even tight - remains open. Further work is also needed to clarify the connections with existing results on degree-2, 3-layered acyclic networks in [4] and the directed cycle networks of [10].

### A. Notation and Basic Terminology

For convenience, the basic notation and terminology used in this paper are described below.

- The set of nonnegative integers (i.e. whole numbers) is denoted  $\mathbb{W}$ .
- A contiguous set  $\{i, i+1, \dots, j\}$  of integers is denoted  $[i : j]$ .
- Random variables (rv's) are written in upper case and their realizations are indicated in corresponding lower case.
- Sets are written in boldface, apart from common ones such as  $\mathbb{R}$  and  $\mathbb{W}$  that are conventionally written in blackboard boldface.
- The set operation  $\mathbf{A} \setminus \mathbf{B}$  denotes  $\mathbf{A} \cap \mathbf{B}^c$ .
- A random signal or process  $(F(k))_{k=0}^{\infty}$  is denoted  $F$ . With a mild abuse of notation, the finite sequence  $(F(k))_{k=0}^t$  is denoted  $F(0 : t)$ .
- Given a subscripted rv or signal  $F_j$ , with  $j$  belonging to a countable set  $\mathbf{J}$ ,  $F_{\mathbf{J}}$  denotes an ordered tuple  $(F_j)_{j \in \mathbf{J}}$ , arranged according to the order on  $\mathbf{J}$ .
- The *entropy* of a discrete-valued rv  $E$  is denoted  $H[E] \geq 0$  and the *differential entropy* of a continuous-valued rv  $F$  (with absolutely continuous distribution) is written  $h[F] \in \mathbb{R}$ . The conditional versions thereof are denoted  $H[E|\cdot]$  and  $h[F|\cdot]$  respectively.
- The *mutual information* in rv's  $E$  and  $F$  is denoted  $I[E; F] \geq 0$ , which is defined as  $H[E] - H[E|F]$  if  $E$  is discrete-valued. If  $E$  is continuous-valued,  $H$  is replaced with the differential entropy  $h$ .
- The *conditional mutual information* in rv's  $E, F$  given  $G$  is denoted  $I[E; F|G]$ , which is defined as  $H[E|G] - H[E|F, G]$  if  $E$  is discrete-valued. If  $E$  is continuous-valued,  $H$  is replaced with  $h$ .
- If  $E, F$  are random processes and  $E$  is discrete-valued, then the *entropy rates* of  $E$ , and the *conditional*

*entropy rate* of  $E$  given (past and present)  $F$  are respectively defined as

$$\begin{aligned} H_\infty[E] &:= \lim_{t \rightarrow \infty} \frac{H[E(0:t)]}{t+1}, \\ H^\infty[E] &:= \overline{\lim}_{t \rightarrow \infty} \frac{H[E(0:t)]}{t+1}, \\ H_\infty[E|F] &:= \lim_{t \rightarrow \infty} \frac{H[E(0:t)|F(0:t)]}{t+1}, \end{aligned}$$

If  $E$  is continuous-valued,  $H$  is simply replaced with  $h$ .

- If  $E, F$  and  $G$  are random processes, then the *mutual information rates* of  $E$  and  $F$ , and the *conditional mutual information rate* of  $E$  and  $F$  given (past and present)  $G$  are respectively defined as

$$\begin{aligned} I^\infty[E;F] &:= \lim_{t \rightarrow \infty} \frac{I[E(0:t);F(0:t)]}{t+1}, \\ I_\infty[E;F] &:= \lim_{t \rightarrow \infty} \frac{I[E(0:t);F(0:t)]}{t+1}, \\ I_\infty[E;F|G] &:= \lim_{t \rightarrow \infty} \frac{I[E(0:t);F(0:t)|G(0:t)]}{t+1}. \end{aligned}$$

- A *directed graph (digraph)*  $(\mathbf{V}, \mathbf{A})$  consists of a set  $\mathbf{V}$  of *vertices*, and a set  $\mathbf{A}$  of *arcs* that each represent a directed link between a particular pair of vertices.
- The initial vertex of an arc is called its *tail* and the terminal vertex, its *head*.
- A *walk* in a digraph is an alternating sequence  $\omega = (v_1, \alpha_1, v_2, \alpha_2, \dots, \alpha_k, v_{k+1})$ ,  $k \geq 0$ , of vertices and arcs, beginning and ending in vertices, s.t. each arc  $\alpha_l$  connects the vertex  $v_l$  to  $v_{l+1}$ . Each vertex  $v_j$  and arc  $\alpha_l$  in the sequence is said to *be in* the walk; with a minor abuse of notation, this is denoted  $v_j \in \omega$ .
- A *path* is a walk that passes through no vertex more than once, including the initial one.
- An *undirected path* is an alternating sequence  $\omega = (v_1, \alpha_1, v_2, \alpha_2, \dots, \alpha_k, v_{k+1})$ ,  $k \geq 0$ , of vertices and arcs, beginning and ending in vertices, s.t. no vertex is repeated and where each arc  $\alpha_l$  connects the vertex  $v_l$  to  $v_{l+1}$ , or  $v_{l+1}$  to  $v_l$ .
- A *cycle* is a walk in which the initial and final vertices are identical, but every other vertex occurs once.
- A *subpath* of a path  $(v_1, \alpha_1, v_2, \alpha_2, \dots, \alpha_k, v_{k+1})$  is a segment  $(v_l, \alpha_l, v_{l+1}, \dots, v_j)$  of it, where  $1 \leq l \leq j \leq k+1$ .
- A vertex  $v$  is said to be *reachable* from another vertex  $\mu$ , denoted  $\mu \rightsquigarrow v$ , if  $\exists$  a path leading from  $\mu$  to  $v$ . Equivalently, it is said that  $\mu$  *can reach*  $v$ . The same terminology and notation apply, with analogous meaning, for pairs of arcs as well as mixed pairs of arcs and vertices. E.g. given an arc  $\beta$ ,  $\mu \rightsquigarrow \beta$  means that there is a path from the vertex  $\mu$  to the tail of  $\beta$ .
- Similarly, a (vertex- or arc-)set  $\mathbf{W}$  is said to be *reachable* from another set  $\mathbf{U}$ , denoted  $\mathbf{U} \rightsquigarrow \mathbf{W}$ , if there is an element of  $\mathbf{W}$  that is reachable from an element of  $\mathbf{U}$ ; equivalently, it is said that  $\mathbf{U}$  *can reach*  $\mathbf{W}$ .
- The notation  $\text{OUT}(\mathbf{U})$  ( $\text{IN}(\mathbf{U})$ ) denotes the set of arcs that have tails (resp. heads) in a vertex-set  $\mathbf{U} \subseteq \mathbf{V}$  but heads (tails)  $\in \mathbf{V} \setminus \mathbf{U}$ . If  $\mathbf{U}$  is a singleton  $\{\mu\}$ , the braces are removed for notational compactness.

## II. PROBLEM FORMULATION

A multiterminal network of unidirectional, point-to-point channels may be modelled using a digraph  $(\mathbf{V}, \mathbf{A})$ , where the vertex-set  $\mathbf{V}$  represents information sources, sinks, repeaters, routers etc., and the arc-set  $\mathbf{A}$  indicates the directions of any channels between network nodes. As usual with digraphs, it is assumed that no arc leaves and enters the same vertex, and that at most one arc leads from the first to the second

element of any given ordered pair of vertices. In other words, every arc in  $\mathbf{A}$  may be uniquely identified with a tuple  $(\mu, \nu) \in \mathbf{V}^2$ , with  $\mu \neq \nu$ .<sup>1</sup> It is also assumed that the digraph is *connected*, i.e. there is an undirected path between any distinct pair of vertices.

In an  $n$ -pairs information network, the locations of sources and sinks are respectively represented by disjoint sets  $\mathbf{S} = \{\sigma_1, \dots, \sigma_n\}$  and  $\mathbf{T} = \{\tau_1, \dots, \tau_n\}$  of distinct vertices in  $\mathbf{V}$ , with each source  $\sigma_i$  aiming to communicate to exactly one sink  $\tau_i$ . It is assumed that  $\sigma_i \rightsquigarrow \tau_i$ . The set of *source-sink pairs*  $(\sigma_i, \tau_i)$ ,  $i \in [1 : n]$ , is denoted  $\mathbf{P} \subset \mathbf{S} \times \mathbf{T}$ . Without loss of generality, it is assumed that every source (sink) has no in-coming (resp. out-going) arcs and exactly one out-going (in-coming) arc.<sup>2</sup> The *boundary*  $\partial\mathbf{V}$  of the network is the set  $\mathbf{S} \cup \mathbf{T}$  of source and sink vertices, and its *interior* is  $\text{int}\mathbf{V} := \mathbf{V} \setminus \partial\mathbf{V}$ .

Each channel in the network can transfer information errorlessly up to a maximum rate, as specified by a positive *arc-capacity*  $c_\alpha \in \mathbb{R}_{>0}$ . In some situations, it may be natural to assign infinite capacity to certain arcs,<sup>3</sup> and the set of all such arcs is denoted  $\mathbf{A}_\infty \subset \mathbf{A}$ . In particular, the arcs leaving sources are by convention assigned infinite capacity. The set of finite-capacity arcs is written  $\mathbf{A}_f = \mathbf{A} \setminus \mathbf{A}_\infty$ , with associated arc-capacity vector  $c := (c_\alpha)_{\alpha \in \mathbf{A}_f} \in \mathbb{R}_{>0}^{|\mathbf{A}_f|}$ . The *structure* of the  $n$ -pairs information network is defined as the tuple  $\Sigma = (\mathbf{V}, \mathbf{A}_f, \mathbf{A}_\infty, \mathbf{P})$ .

The communication signals in the channels are represented by a vector  $S \equiv (S_\alpha)_{\alpha \in \mathbf{A}}$  of random processes called *arc-signals*, with  $S_\alpha : \mathbb{W} \rightarrow \mathbb{R}^{m_\alpha}$  taken to be discrete-valued  $\forall \alpha \in \mathbf{A}_f$ . In particular, the arc-signals leaving sources and entering sinks respectively represent the exogeneous inputs and outputs of the information network. For convenience, the signal  $S_{\text{OUT}(\sigma_i)}$  leaving the  $i$ -th source  $\sigma_i \in \mathbf{S}$  is denoted  $X_i$ , and the signal  $S_{\text{IN}(\tau_i)}$  entering the  $i$ -th sink  $\tau_i \in \mathbf{T}$  is called  $Y_i$ . It is assumed throughout this paper that the signals  $X_1, X_2, \dots, X_n$  are mutually independent, though each may itself be a correlated process, and that the signals on finite-capacity arcs are discrete-valued.

The arc-signal vector  $S$  is assumed to have the following property:

**Definition 1 (Setwise Causality and Signal Graphs):** An arc-signal vector  $S$  is called *setwise causal* on a structure  $\Sigma = (\mathbf{V}, \mathbf{A}_f, \mathbf{A}_\infty, \mathbf{P})$  if all arc-signals leaving any internal vertex-set  $\mathbf{U} \subseteq \text{int}\mathbf{V}$  are causally determined by those entering it. That is,  $\forall \mathbf{U} \subseteq \text{int}\mathbf{V}$ ,  $\exists$  an operator  $g_{\mathbf{U}}$  s.t.

$$S_{\text{OUT}(\mathbf{U})}(t) = g_{\mathbf{U}}(t, S_{\text{IN}(\mathbf{U})}(0 : t)), \quad \forall t \in \mathbb{W}. \quad (1)$$

The tuple  $(\Sigma, S)$  is then called a *signal graph*.  $\diamond$

**Remarks:** Setwise causality is a generalization of the fundamental concept of *well-posedness* in feed-back control theory. In acyclic digraphs (i.e. in which every walk is a path), it is equivalent to causality at every internal vertex. However, feedback signals may be present in cyclic digraphs, in which case vertex-wise causality would not suffice to guarantee that outgoing signals from an arbitrary vertex-set are uniquely and causally determined by incoming ones as required in (1). Stronger assumptions would be needed, e.g. a positive time-delay at every vertex.

In an  $n$ -pairs network, information must be conveyed from each source  $\sigma_i \in \mathbf{S}$  to its corresponding sink  $\tau_i$  so as to achieve some goal, without exceeding any channel capacities. In this paper, the goal is to guarantee that the marginal information rates supplied to the sinks exceed specified *demands*. This leads to the following definition:

**Definition 2 (Achievability):** Consider an  $n$ -pairs information network with structure  $\Sigma$ , source-signal vector  $X$ , arc-capacity vector  $c \in \mathbb{R}_{>0}^{|\mathbf{A}_f|}$  and demand vector  $d := (d_i)_{i=1}^n \in \mathbb{R}_{>0}^n$ . The tuple  $(\Sigma, X, c, d)$  is called *achievable* if  $\exists$  a setwise-causal (Def. 1) arc-signal vector  $S$  s.t.

$$S_{\text{OUT}(\sigma_i)} = X_i, \quad \forall i \in [1 : n], \quad (2)$$

$$\mathbf{I}_\infty[Y_i; X_i] \geq d_i, \quad \forall i \in [1 : n], \quad (3)$$

$$\mathbf{I}^\infty[S_\alpha; X] \leq c_\alpha, \quad \forall \alpha \in \mathbf{A}_f. \quad (4)$$

<sup>1</sup>Such digraphs are sometimes called *simple*.

<sup>2</sup>If a source or sink were actually connected to multiple nodes in the network, it would be represented in the digraph by an auxiliary vertex connected by an arc (of infinite capacity) with a multiply-connected vertex.

<sup>3</sup>For instance, when a single network node is represented as two “virtual” vertices connected by an arc of unbounded capacity.

Such an  $S$  is called a *solution* to the  $n$ -pairs information network problem  $(\Sigma, X, c, d)$ . The demand vector  $d$  is then called achievable on  $(\Sigma, X, c)$ ;  $c$  is called achievable on  $(\Sigma, X, d)$ ; and  $(X, c, d)$  is called achievable on  $\Sigma$ .

The closure of the set of achievable demand vectors  $d$  is called the *demand region*  $\mathbf{D} \subseteq \mathbb{R}_{\geq 0}^n$  of  $(\Sigma, X, c)$ .

◇

**Remarks:** The objective (3) is a relaxation of the usual aim of reconstructing the source signals reliably or within some specified distortion level. This allows the standard assumption that each source signal is stationary or i.i.d. to be dropped. Note that when  $X_i$  is discrete-valued,  $d_i = H_\infty[X_i]$  is necessary for reliable reconstruction at  $\tau_i$ .

Also note, it is implicit in Def. 2 that the input-output operators  $g_v$  at every internal vertex  $v$  may be freely designed to yield a solution  $S$ , as long as setwise causality (Def. 1) is respected.

As mentioned in the introduction, it was once thought that a network was achievable<sup>4</sup> iff it admitted a *routing* solution. In the present context, this is equivalent to presuming the existence of an  $(X, c, d)$ -feasible multicommodity flow, i.e. of a nonnegative tuple  $f = (f_{\alpha,j})_{\alpha \in \mathbf{A}, j \in [1:n]} \in \mathbb{R}_{\geq 0}^{|\mathbf{A}|n}$ , of bit-rates on each arc associated with every source-sink pair, s.t.

$$\sum_{j=1}^n f_{\alpha,j} \leq c_\alpha, \quad \forall \alpha \in \mathbf{A}_f \quad (\text{capacity bound}); \quad (5)$$

$$f_{\text{IN}(\tau_j),j} \geq d_j, \quad \forall j \in [1:n] \quad (\text{satisfaction of demand}), \quad (6)$$

$$f_{\text{OUT}(\sigma_j),j} \leq H_\infty[X_j], \quad \forall j \in [1:n] \quad (\text{sufficiency of supply}), \quad (7)$$

$$\sum_{\alpha \in \text{IN}(v)} f_{\alpha,j} = \sum_{\alpha \in \text{OUT}(v)} f_{\alpha,j} \quad (\text{conservation of flow}), \quad (8)$$

for any  $j \in [1:n]$  and  $v \in \mathbf{V} \setminus (\{\sigma_j\} \cup \{\tau_j\})$ . Via an explicit counter-example, the article [3] showed that this intuitive notion was incorrect, i.e. that although the existence of a feasible multicommodity flow is sufficient for achievability, it is not generally necessary. This laid the foundations for *network coding*, in which nodes are permitted to not just route incoming bits, but also to perform functions on them.

Nonetheless, routing/multicommodity-flow solutions have certain virtues, as discussed in sec. I. This paper poses the question: is there a general class of  $n$ -pair information network structures  $\Sigma$  for which the achievability of  $(X, c, d)$  is equivalent to the existence of an  $(X, c, d)$ -feasible multicommodity flow  $f$  (5)–(8)?

Any  $n$ -pairs information network structure  $\Sigma$  can support  $(X, c, d)$ -feasible multicommodity flows if the arc-capacities and source entropy rates are sufficiently larger than the demands (and provided that each sink is reachable from its source). However, there are examples of structures on which an  $(X, c, d)$ -feasible multicommodity flow does not exist if arc-capacities are reduced or demands increased, even though  $(X, c, d)$  is still achievable (see sec. V).

The aim of this paper is to isolate certain structural properties that ensure routability *over all achievable combinations of*  $(X, c, d)$ . Such properties would inhere solely in  $\Sigma$ , suiting situations in which channels, switches, transceivers and interfaces are expensive to set up and difficult to move, and/or where channel capacities, output demands and source-signal statistics are variable or unknown.

### III. MAIN RESULT AND REQUISITE GRAPH-THEORETIC CONCEPTS

Before giving the (affirmative) answer to the question above, several graph-theoretic notions must be defined for an  $n$ -pairs information network with structure  $\Sigma = (\mathbf{V}, \mathbf{A}_f, \mathbf{A}_\infty, \mathbf{P})$ .

First, some largely familiar concepts are revisited. A path in an  $n$ -pairs information network that goes from a source  $\sigma_i$  to its sink  $\tau_i$  is called an *i-path* and the set of all *i*-paths is called an *i-bundle*. In other words, the *i*-bundle is the set of all acyclic paths via which information can be routed from  $\sigma_i$  to  $\tau_i$ .

<sup>4</sup>ignoring differences in the definition of achievability

Given a set  $\mathbf{J} \subseteq [1 : n]$ , the set of all  $i$ -paths with  $i \in \mathbf{J}$  is called a **J-bundle** (not the same as the set of all  $\{\sigma_i : i \in \mathbf{J}\} \rightsquigarrow \{\tau_i : i \in \mathbf{J}\}$ -paths, which contains it). Let  $(\mathbf{V}^i, \mathbf{A}^i)$  denote the subgraph formed by all the vertices and arcs in the  $i$ -bundle. A vertex-set  $\mathbf{U} \subset \mathbf{V}^i$  such that  $\sigma_i \in \mathbf{U}$  and  $\tau_i \notin \mathbf{U}$  is called an  $i$ -cut.

The following notions are somewhat nonstandard.

**Definition 3 (J-Disjointness):** Given a structure  $\Sigma$ , an arc-set  $\mathbf{B} \subseteq \mathbf{A}$  is called **J-disjoint** for an index set  $\mathbf{J} \subseteq [1 : n]$  if each path in the **J-bundle** passes through at most one arc in  $\mathbf{B}$ .  $\diamond$

**Remarks:** Every singleton  $\{\alpha\} \subseteq \mathbf{A}$  is automatically **J-disjoint** and every arc-set  $\mathbf{B} \subseteq \mathbf{A}$  is automatically  $\emptyset$ -disjoint.

**Definition 4 (Chokes, Wholeness and Triangularity):** Given a structure  $\Sigma$  and vertex-sets  $\mathbf{W}, \mathbf{Z} \subseteq \mathbf{V}$ , an arc-set  $\mathbf{B} \subseteq \mathbf{A}$  is called a **W  $\rightsquigarrow$  Z-choke** if all paths from the vertex-set  $\mathbf{W}$  to  $\mathbf{Z}$  pass through  $\mathbf{B}$ .

Given a source index  $i \in [1 : n]$ , a  $\sigma_i \rightsquigarrow \tau_i$ -choke is also called an  $i$ -choke.

An  $i$ -choke  $\mathbf{B}$  is further described as **downward-whole** if all  $\sigma_j \rightsquigarrow \tau_i$ -paths pass through it,  $\forall j \in [1 : i - 1]$  s.t.  $\sigma_j \rightsquigarrow \mathbf{B}$ .

An arc-set  $\mathbf{B} \subseteq \mathbf{A}$  is called a  $[1 : i]$ -choke if it is a  $j$ -choke,  $\forall j \in [1 : i]$ .

A  $[1 : i]$ -choke is called **triangular** if it is a downward-whole  $j$ -choke,  $\forall j \in [1 : i]$ .

$\diamond$

**Remarks:** Downward-whole  $i$ -chokes are useful because they are “immune” to any information about  $X_1, \dots, X_{i-1}$  that may be transmitted to the sink  $\tau_i$  without having passed through them, and thus obey a version of the data-processing inequality (see Lemma 4 in the appendices). Their immunity arises because  $\forall k \in [1 : i - 1]$ , either no  $\sigma_k \rightsquigarrow \tau_i$ -walk passes through them, or all of them do.

Note also that a (triangular)  $[1 : i]$ -choke is automatically a (triangular)  $[1 : i - 1]$ -choke.

**Definition 5 (Viable i-Cuts):** Given an index  $i \in [1 : n]$  and a structure  $\Sigma$ , an  $i$ -cut  $\mathbf{U} \subset \mathbf{V}^i$  is called **viable** under the following conditions:

- 1) Every arc in  $\text{OUT}(\mathbf{U}) \cap \mathbf{A}^i$  is finite-capacity.
- 2) There is an  $i$ -path that leaves  $\mathbf{U}$  without re-entering.
- 3) Each arc in  $\text{OUT}(\mathbf{U}) \cap \mathbf{A}^i$  lies in an  $i$ -path that exits  $\mathbf{U}$  without re-entering or that lies in the  $[1 : i - 1]$ -bundle.
- 4) Every vertex  $v \in \mathbf{U}$  lies on an undirected path  $\pi$  from  $\sigma_i$  to  $v$  such that
  - a) all vertices before  $v$  on  $\pi$  are in  $\mathbf{U}$ , and
  - b) every reverse-oriented arc in  $\pi$  (i.e. pointing from  $v$  to  $\sigma_i$ ) lies on an  $i$ -path that does not re-enter  $\mathbf{U}$ .

$\diamond$

**Remark:** These are the structural properties of min-cuts in a residual capacitated digraph that is used in appendix C to prove the main result.

**Definition 6 (Reverse Structure):** Given a structure  $\Sigma$ , the **reverse structure**  $\Sigma' := (\mathbf{V}, \mathbf{A}'_f, \mathbf{A}'_\infty, \mathbf{P}')$  is given by

$$\mathbf{A}'_f := \{(\mu', v') : (v', \mu') \in \mathbf{A}_f\}, \quad (9)$$

$$\mathbf{A}'_\infty := \{(\mu', v') : (v', \mu') \in \mathbf{A}_\infty\}, \quad (10)$$

$$\mathbf{P}' := \{(\sigma'_i, \tau'_i) : (\tau'_i, \sigma'_i) \in \mathbf{P}\}. \quad \diamond \quad (11)$$

**Remarks:** This describes the  $n$ -pairs information network obtained by reversing arcs and swapping the roles of sources and sinks. The reverse structure is useful because a multicommodity flow on  $\Sigma$  can be reversed in direction to yield one on  $\Sigma'$  and vice-versa.

**Definition 7 (Triangularizability):** A structure  $\Sigma$  is called **triangularizable** if there is an ordering  $(\sigma_1, \delta_1), \dots, (\sigma_n, \delta_n)$  of the source-sink pairs so that  $\forall i \in [2 : n]$  and every viable  $i$ -cut  $\mathbf{U}$  (Def. 5), a  $[1 : i - 1]$ -disjoint, triangular  $[1 : i]$ -choke is contained in  $\text{OUT}(\mathbf{U}) \cap \mathbf{A}^i$ . (Def. 4).  $\diamond$

**Remark:** Triangularizability is not generally invariant under reversals of structure.

The main result of this paper can now be stated:

**Theorem 1 (Triangularizability Implies Structural Routability):** Suppose an  $n$ -pairs information network has a structure  $\Sigma$  or reverse structure  $\Sigma'$  (Def. 6) that is triangularizable (Def. 7). Then  $\Sigma$  is structurally routable, i.e.  $(\Sigma, X, c, d)$  is achievable (Def. 2) iff an  $(X, c, d)$ -feasible multicommodity flow  $f$  (5)–(8) exists on  $\Sigma$ .  $\nabla$

The proof of this result is given in the appendices, but for convenience its central ideas are described in the next section. In sec. V, two network examples are discussed to illustrate Thm. 1.

#### IV. KEY ELEMENTS IN THE PROOF OF THM. 1

Throughout this section,  $\Sigma = (\mathbf{V}, \mathbf{A}_f, \mathbf{A}_\infty, \mathbf{P})$  is the structure of an  $n$ -pairs information network, as described in sec. II. In both the proofs of necessity and sufficiency, use will be made of the fact that  $\forall i \in [1 : n]$ , any single-commodity flow  $q$  from  $\sigma_i$  to  $\tau_i$  in the structure  $\Sigma$  can be decomposed into a superposition of *i-path flows* and *cycle flows* (see e.g. [11], Thm. 3.3.1). That is, if  $\pi_{1,i}, \dots, \pi_{p_i,i}$  are the distinct  $i$ -paths and  $\gamma_1, \dots, \gamma_g$ , the distinct cycles, then  $\exists$  numbers  $u_{1,i}, \dots, u_{p,i} \geq 0$  and  $w_{1,i}, \dots, w_{g,i} \geq 0$  s.t.

$$q_\alpha = \sum_{1 \leq k \leq p_i : \pi_{k,i} \ni \alpha} u_{k,i} + \sum_{1 \leq l \leq g : \gamma_l \ni \alpha} w_{l,i}. \quad (12)$$

If  $w_{l,i} = 0$  for all  $l \in [1 : g]$ , then  $q$  is called *acyclic*.

The proof of sufficiency in app. D is relatively straightforward. If  $f$  is an  $(X, c, d)$ -feasible multicommodity flow (5)–(8) on  $\Sigma$ , the decomposition (12) is used directly to devise a routing solution  $S$ .

The proof of necessity in app. C is more difficult and involves induction, using the following building blocks.

**Definition 8 (J-Flow):** Given an index set  $\mathbf{J} \subseteq [1 : n]$ , a nonnegative tuple  $f = (f_{\alpha,j})_{\alpha \in \mathbf{A}, j \in \mathbf{J}} \in \mathbb{R}_{\geq 0}^{|\mathbf{A}||\mathbf{J}|}$  is called a **J-flow** on the structure  $\Sigma$  if  $\forall j \in \mathbf{J}$  and  $v \in \mathbf{V} \setminus \{\sigma_j\} \cup \{\tau_j\}$ ,

$$\sum_{\alpha \in \text{IN}(v)} f_{\alpha,j} = \sum_{\alpha \in \text{OUT}(v)} f_{\alpha,j} \quad (j\text{-flow conservation}), \quad (13)$$

As a convention, the  $\emptyset$ -flow is defined as the empty sequence  $()$ .  $\diamond$

**Remark:** A **J-flow** is a (possibly infeasible) multicommodity flow with source-sink pairs  $(\sigma_j, \tau_j)$ ,  $j \in \mathbf{J}$ . If each  $j$ -flow  $f_{\mathbf{A},j}$  is acyclic,  $\forall j \in \mathbf{J}$ , then  $f$  is called an *acyclic J-flow*.

The next concept is central to the proof of necessity. It defines a class of feasible  $[1 : i]$ -flows that obey certain information-theoretic bounds when only the signals  $X_j$ ,  $j \in [1 : i]$  need to be communicated.

**Definition 9 (Informational Feasibility):** Given  $i \in [1 : n]$  and a setwise-causal arc-signal vector  $S$  (Def. 1), a  $[1 : i]$ -flow  $f \in \mathbb{R}_{\geq 0}^{|\mathbf{A}|i}$  (Def. 8) is called *informationally feasible* on the signal graph  $(\Sigma, S)$  if it satisfies the following conditions:

i) On every arc  $\alpha \in \mathbf{A}_f$ ,

$$\sum_{j=1}^i f_{j,\alpha} \leq c_\alpha. \quad (14)$$

ii) On any  $[1 : i]$ -disjoint, triangular  $[1 : i]$ -choke  $\mathbf{B}$  (Defs. 3 & 4),

$$\sum_{\alpha \in \mathbf{B}, j \in [1:i]} f_{\alpha,j} \leq I_\infty [X_{[1:i]}; S_{\mathbf{B}} | X_{[i+1:n]}]. \quad (15)$$

iii) On arcs entering sinks and leaving sources,

$$f_{\text{IN}(\tau_j),j} = f_{\text{OUT}(\sigma_j),j} = I_\infty [X_j; Y_j | X_{[j+1:n]}], \quad \forall j \in [1 : i]. \quad (16)$$

$\diamond$

**Remarks:** Intuitively, (15) is restricted to  $[1 : i]$ -disjoint sets because the flow contribution corresponding to a path in the  $[1 : i]$ -bundle could otherwise be counted more than once. Note as well that the  $\emptyset$ -flow is

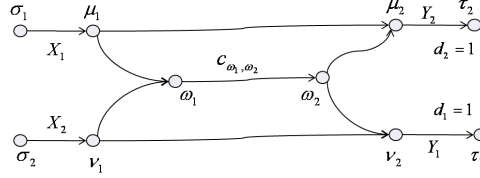


Fig. 1. Butterfly Network

informationally feasible, since the condition (16) disappears and the inequality (15) has a zero left-hand side (LHS) and so is trivially satisfied.

The following result concerning informationally feasible  $[1 : n]$ -flows also plays a key role.

*Lemma 1:* If the arc-signal vector  $S$  is a solution to the  $n$ -pairs information network problem  $(\Sigma, X, c, d)$  (Def. 2), then any informationally feasible (Def. 9)  $[1 : n]$ -flow  $f \in \mathbb{R}_{\geq 0}^{|\mathbf{A}|^n}$  on  $(\Sigma, S)$  is an  $(X, c, d)$ -feasible multicommodity flow on  $\Sigma$  (5)–(8).  $\nabla$

*Proof:* See appendix B.

The proof of necessity in appendix C uses upward induction to construct a sequence of informationally feasible  $[1 : i]$ -flows (Def. 8),  $i \in [1 : n]$ . By Lemma 1,  $f^n$  is then an  $(X, c, d)$ -feasible multicommodity flow, as desired.

## V. EXAMPLES

The first example, a 2-pairs butterfly network, is adapted from [2], [12] and depicted in Fig. 1. All its arcs have infinite capacities except on the arc from  $\omega_1$  to  $\omega_2$ . The signals  $X_1$  and  $X_2$  are i.i.d. and binary-valued with  $H[X_1(t)] = H[X_2(t)] = 1$ , and the demands  $d_1 = d_2 = 1$ .

The vertex-set  $\mathbf{U} = \{\sigma_2, v_1, \omega_1\} \subset \mathbf{V}_2 = \{\sigma_2, v_1, \omega_1, \omega_2, \mu_2, \tau_2\}$  is a 2-cut. Observe that

- 1)  $\text{OUT}(\mathbf{U}) \cap \mathbf{A}^2 = \{(\omega_1, \omega_2)\}$  is finite-capacity,
- 2) the 2-path  $(\sigma_2, v_1, \omega_1, \omega_2, \mu_2, \tau_2)$  leaves  $\mathbf{U}$  without returning,
- 3)  $\text{OUT}(\mathbf{U}) \cap \mathbf{A}^2 = \{(\omega_1, \omega_2)\}$  lies in this non-returning 2-path, and
- 4) every vertex  $\in \mathbf{U}$  lies in an undirected path from  $\sigma_2$  such that all its predecessors in the path are also in  $\mathbf{U}$ , with no reverse-oriented arcs.

Consequently  $\mathbf{U}$  is a viable 2-cut (Def. 5). However,  $\text{OUT}(\mathbf{U}) \cap \mathbf{A}^2 = \{(\omega_1, \omega_2)\}$  does not contain any triangular  $[1 : 2]$ -choke (Def. 4), since one  $\sigma_1 \rightsquigarrow \tau_2$ -path,  $(\sigma_1, \mu_1, \mu_2, \tau_2)$ , bypasses it but the other,  $(\sigma_1, \mu_1, \omega_1, \omega_2, \mu_2, \tau_2)$ , does not. Consequently, this network is not triangularizable (Def. 7) and Thm. 1 does not apply.

This is in agreement with [12]. Observe that if  $c \equiv c_{\omega_1, \omega_2} = 2$ , then  $(X, c, d)$  is achievable by routing, since 1 unit of flow can be routed from  $\sigma_1$  to  $\tau_1$  and from  $\sigma_2$  to  $\tau_2$  via  $(\omega_1, \omega_2)$ , without exceeding its arc-capacity. However, if  $c$  is reduced to 1, then the network can no longer support an  $(X, c, d)$ -feasible 2-commodity flow (5)–(8). Nonetheless,  $(X, c, d)$  still remains achievable, by sending the signal  $X_1$  on  $(\mu_1, \mu_2)$ ,  $X_2$  on  $(v_1, v_2)$ , and  $Z = X_1 + X_2 \pmod{2}$  on  $(\omega_1, \omega_2)$ ,  $(\omega_2, \mu_2)$  and  $(\omega_2, v_2)$ , and setting  $Y_2 = Z + X_1 \pmod{2} = X_2$  and  $Y_1 = Z + X_2 \pmod{2} = X_1$ .

Next, consider the cyclic 2-pairs network of Fig. 2. The only 2-cuts  $\mathbf{U}^k$  with  $\text{OUT}(\mathbf{U}^k) \cap \mathbf{A}^2 \subseteq \mathbf{A}_f$  are  $\mathbf{U}^1 = \{\sigma_2, \mu_1, \omega\}$ ,  $\mathbf{U}^2 = \{\sigma_2, \mu_1, \omega, v_1\}$  and  $\mathbf{U}^3 = \{\sigma_2, \mu_1, \omega, v_1, v_2\}$ . For the first two 2-cuts,

$$\text{OUT}(\mathbf{U}^k) \cap \mathbf{A}^2 = \{(\omega, \mu_2), (\mu_1, v_2)\}, \quad k = 1, 2,$$

while  $\text{OUT}(\mathbf{U}^3) \cap \mathbf{A}^2 = \{(\omega, \mu_2)\}$ . Observe that

- There is a 2-path,  $\pi = (\sigma_2, \mu_1, \omega, \mu_2, \tau_2)$ , that leaves each  $\mathbf{U}^k$  without re-entering.
- The single arc  $(\omega, \mu_2)$  in  $\text{OUT}(\mathbf{U}^3) \cap \mathbf{A}^2$  lies in the non-returning 2-path  $\pi$ . The other arc  $(\mu_1, v_2)$  in  $\text{OUT}(\mathbf{U}^k) \cap \mathbf{A}^2$ ,  $k = 1, 2$ , does not lie on a non-returning 2-path, but does lie on a 1-path.
- Every vertex in  $\mathbf{U}^1$  lies in an undirected path from  $\sigma_2$  that does not leave  $\mathbf{U}^1$  and does not contain any backward-oriented arcs; the same for  $\mathbf{U}^3$ .



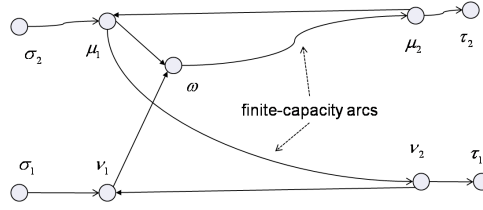


Fig. 2. Cyclic Network

- Every vertex  $v$  in  $\mathbf{U}^2$  lies in one undirected path from  $\sigma_2$  that does not leave  $\mathbf{U}^2$ . However, when  $v = \nu_1$ , this path contains a backward arc  $(\nu_1, \omega)$  that does not lie on a non-returning 2-path.

Consequently, the only viable 2-cuts (Def. 5) are  $\mathbf{U}^1$  and  $\mathbf{U}^3$ .

Now, each arc-set  $\text{OUT}(\mathbf{U}^k) \cap \mathbf{A}^2$ ,  $k = 1, 3$ , contains an arc-set  $\mathbf{B} = \{(\omega, \mu_2)\}$ , which being a singleton is automatically 1-disjoint. Furthermore, both 2-paths and the only 1-path pass through it, as does the only  $\sigma_1 \rightsquigarrow \tau_2$ -path. Thus  $\mathbf{B}$  is a 1-disjoint, triangular  $[1 : 2]$ -choke, and so the network is triangularizable (Def. 7). Consequently, by Thm. 1 any tuple  $(X, c, d)$  is achievable iff there exists an  $(X, c, d)$ -feasible multicommodity flow. After eliminating the flow variables,  $(X, c, d)$  is thus achievable iff

$$\begin{aligned} d_1 + d_1 &\leq c_{\omega, \mu_2}, \\ d_1 &\leq \min \{H_\infty[X_1], c_{\mu_1, \nu_2}\}, \\ d_2 &\leq H_\infty[X_2]. \end{aligned}$$

## VI. CONCLUSION

This paper examined the routability of possibly cyclic  $n$ -pairs information networks from a structural perspective. The concept of triangularizability was introduced, and it was shown that for networks with triangularizable structures, routability and achievability are equivalent, i.e. a given combination of source signals, demand rates and channel capacities is achievable iff the network supports a feasible multicommodity flow.

Triangularizability is a conservative structural condition, and future work will focus on trying to relax it. The inductive nature of the proof of necessity in this paper requires it directly, so such a relaxation may entail a very different analysis. As a first step, more work must be done to clarify any connections to the examples in the literature [4], [10] of special network structures for which routing always achieves capacity.

## ACKNOWLEDGEMENTS

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## APPENDIX A

### SOME TECHNICAL LEMMAS

Throughout this sub-subsection,  $\Sigma = (\mathbf{V}, \mathbf{A}_f, \mathbf{A}_\infty, \mathbf{P})$  is the structure of an  $n$ -pairs information network and  $\Gamma = (\Sigma, S)$ , its signal graph, as described in sec. II and Def. 1.

The following lemmas are intuitive, though some care is required in the proofs to construct suitable internal vertex-sets for applying setwise causality (Def. 1).

**Lemma 2:** In the signal graph  $\Gamma$ , the arc-signal  $S_\alpha$  on any arc  $\alpha \in \mathbf{A}$  is causally determined by the source-signal vector  $X_{\mathbf{K}}$ , where  $\mathbf{K} \subseteq [1 : n]$  is the index-set of all sources that can reach  $\alpha$ .  $\nabla$

*Proof:* Let  $\mathbf{U} \subseteq \text{int}\mathbf{V}$  be the set of all internal vertices that can reach  $\alpha$ . Observe that the unique tail  $\mu \in \mathbf{V}$  of any  $\beta \in \text{IN}(\mathbf{U})$  can automatically reach  $\alpha$  and thus would be  $\in \mathbf{U}$  if it were internal. However, as  $\mu$  is outside  $\mathbf{U}$ , by definition of  $\text{IN}(\mathbf{U})$ , it must therefore be a boundary vertex and specifically a source  $\sigma_i$  with  $i \in \mathbf{K}$ . As  $\alpha \in \text{OUT}(\mathbf{U})$ , the proof then follows from setwise causality (Defn. 1).  $\square$

**Lemma 3:** Let  $\mathbf{B} \subseteq \mathbf{A}$  be a  $\{\sigma_j : j \in \mathbf{J}\} \rightsquigarrow \tau_i$ -choke (Def. 4) in the signal graph  $\Gamma$ , where  $\mathbf{J} \subseteq [1 : n]$ . Then the arc-signal  $Y_i$  entering the sink  $\tau_i$  is causally determined by the arc-signal vector  $S_{\mathbf{B}}$  on the choke and the source-signal vector  $X_{\mathbf{J}^c}$ , where  $\mathbf{J}^c = [1 : n] \setminus \mathbf{J}$ .  $\nabla$

*Proof:* Define  $\mathbf{S}_{\mathbf{J}} := \{\sigma_j : j \in \mathbf{J}\}$  and  $\mathbf{S}_{\mathbf{J}^c} := \{\sigma_j : j \in \mathbf{J}^c\}$ , and let  $\pi_1, \dots, \pi_l$  be the different paths from  $\mathbf{S}_{\mathbf{J}}$  to  $\tau_i$ , each of which must pass through  $\mathbf{B}$ . Let  $\gamma_k$  be the last arc  $\in \mathbf{B}$  on  $\pi_k$ ;  $\rho_k$ , the subpath along  $\pi_k$  from the head of  $\gamma_k$  to  $\tau_i$ ; and  $\mathbf{V}_k$ , the set of all vertices  $\neq \tau_i$  (therefore internal) on  $\rho_k$  that are traversed after the arc  $\gamma_k$ . Define  $\mathbf{U} := \bigcup_{k=1}^l \mathbf{V}_k$  and  $\mathbf{C} := \{\gamma_k : 1 \leq k \leq l\} \subseteq \mathbf{B}$ . Let  $\mathbf{D} := \mathbf{B} \cap \text{IN}(\mathbf{U})$  and  $\mathbf{E} := \mathbf{B}^c \cap \text{IN}(\mathbf{U})$ .

It is next shown that no arc  $\eta \in \mathbf{E}$  is reachable from  $\mathbf{S}_{\mathbf{J}}$ . Let  $\eta$  have tail  $v$  and head  $\mu$ . Note that  $v \notin \mathbf{U}$  and  $\mu \in \mathbf{U}$ , by definition of  $\text{IN}(\mathbf{U})$ . Plainly,  $\mu$  must be the first vertex of a (possibly nonunique) path  $\rho_k$  leading to  $\tau_i$ . In addition,  $\mu$  has an incoming arc  $\gamma_k \in \mathbf{D}$ . Suppose that there were to be a path  $\pi$  from  $\mathbf{S}_{\mathbf{J}}$  to  $v$ . There would then be a path  $(\pi, \eta, \rho_k) \equiv \pi_m$  from  $\mathbf{S}_{\mathbf{J}}$  to  $\tau_i$ , via the arc  $\eta \notin \mathbf{B}$ . By definition, this path would have to pass through  $\mathbf{B}$  at some stage, with the last arc of  $\mathbf{B}$  it passes through denoted  $\gamma_m$ . As  $\eta \notin \mathbf{B}$  and  $\rho_k$  does not pass through  $\mathbf{B}$ , it follows that  $\gamma_m$  lies on  $\pi$ . Thus all non- $\tau_i$  vertices, including  $v$  that are after the arc  $\gamma_m$  on  $\pi_m$ , would have to be in  $\mathbf{U}$ , contradicting the fact that  $v \notin \mathbf{U}$ .

As the arc leading into  $\tau_i$  is an element of  $\text{OUT}(\mathbf{U})$ , set-wise causality (1) implies that

$$Y_i(t) \equiv S_{\text{IN}(\tau_i)}(t) \equiv g'_{\mathbf{U}}(t, S_{\mathbf{D}}(0:t), S_{\mathbf{E}}(0:t)), \quad \forall t \in \mathbb{W}. \quad (17)$$

As  $\mathbf{E}$  is unreachable from  $\mathbf{S}_{\mathbf{J}}$ , Lemma 2 implies that

$$S_{\mathbf{E}}(t) \equiv g_{\mathbf{U}}^*(t, X_{\mathbf{J}^c}(0:t)), \quad \forall t \in \mathbb{W},$$

which substituted into (17) yields

$$\begin{aligned} Y_i(t) &= g'_{\mathbf{U}}(t, S_{\mathbf{D}}(0:t), g_{\mathbf{U}}^*(t, X_{\mathbf{J}^c}(0:t))) \\ &\equiv g''_{\mathbf{U}}(t, S_{\mathbf{D}}(0:t), X_{\mathbf{J}^c}(0:t)), \quad \forall t \in \mathbb{W}. \end{aligned}$$

Observing that  $\mathbf{D} \subseteq \mathbf{B}$  completes the proof.  $\square$

**Lemma 4:** If  $\mathbf{B} \subseteq \mathbf{A}$  is a downward-whole  $i$ -choke (Def. 4), then

$$\mathbf{I}_\infty [X_i; Y_i | X_{[i+1:n]}] \leq \mathbf{I}_\infty [X_i; S_{\mathbf{B}} | X_{[i+1:n]}]. \quad \nabla$$

*Proof:* Let  $\mathbf{K}$  be the set of all source indices  $j \in [1 : n]$  such that  $\sigma_j \rightsquigarrow \mathbf{B}$ , and define  $\mathbf{J} := \mathbf{K} \cap [1 : i]$  and  $\mathbf{J}' := [1 : i] \setminus \mathbf{J}$ . From Def. 4,  $\mathbf{J} \ni i$  and all  $\{\sigma_j : j \in \mathbf{J}\} \rightsquigarrow \tau_i$ -paths pass through  $\mathbf{B}$ . By Lemma 3,  $Y_i$  is thus causally determined by  $S_{\mathbf{B}}$  and  $X_{\mathbf{J}^c} \equiv (X_{\mathbf{J}'}, X_{[i+1:n]})$ , i.e.

$$Y_i(0:t) \equiv g(t, S_{\mathbf{B}}(0:t), X_{\mathbf{J}'}(0:t), X_{[i+1:n]}(0:t)), \quad \forall t \in \mathbb{W},$$

where  $g$  is a deterministic, time-varying mapping. Furthermore, by Lemma 2,  $S_{\mathbf{B}}$  is causally determined by  $X_{\mathbf{K}}$ , and hence by  $X_{\mathbf{J}}$  and  $X_{[i+1:n]}$ . So the argument  $X_{\mathbf{J}'}(0:t)$  in the RHS of the equation above must be independent of  $(S_{\mathbf{B}}(0:t), X_{[i+1:n]}(0:t), X_i(0:t))$ . From this, it follows that that

$$X_i(0:t) \leftrightarrow (S_{\mathbf{B}}(0:t), X_{[i+1:n]}(0:t)) \leftrightarrow (Y_i(0:t), X_{[i+1:n]}(0:t))$$

forms a Markov chain. By the data processing inequality,

$$\begin{aligned} \mathbf{I}[X_i(0:t); S_{\mathbf{B}}(0:t) | X_{[i+1:n]}(0:t)] &= \mathbf{I}[X_i(0:t); S_{\mathbf{B}}(0:t), X_{[i+1:n]}(0:t)] \\ &\geq \mathbf{I}[X_i(0:t); Y_i(0:t), X_{[i+1:n]}(0:t)] \\ &= \mathbf{I}[X_i(0:t); Y_i(0:t) | X_{[i+1:n]}(0:t)], \end{aligned}$$

where the equalities are due to the mutual independence of the source signals. Dividing by  $t+1$  and taking inferior limits completes the proof.  $\square$

## APPENDIX B PROOF OF LEMMA 1

It will be shown that (13) with  $\mathbf{J} = [1 : n]$ , (14)–(16) with  $i = n$ , and Def. 2 imply (5)–(8). First, observe that (13) with  $\mathbf{J} = [1 : n]$  is identical to (8), and (14) with  $i = n$ , to (5). Next, note that (16) with  $i = n$  yields

$$\begin{aligned} f_{\text{IN}(\tau_j), j} &= \mathbf{I}_{\infty}[X_j; Y_j | X_{[j+1:n]}] \\ &\equiv \lim_{t \rightarrow \infty} \left\{ \frac{\mathbf{H}[X_j(0:t) | X_{[j+1:n]}(0:t)]}{t+1} \right. \\ &\quad \left. - \frac{\mathbf{H}[X_j(0:t) | Y_j(0:t), X_{[j+1:n]}(0:t)]}{t+1} \right\} \\ &= \lim_{t \rightarrow \infty} \frac{\mathbf{H}[X_j(0:t)] - \mathbf{H}[X_j(0:t) | Y_j(0:t), X_{[j+1:n]}(0:t)]}{t+1} \end{aligned} \tag{18}$$

$$\begin{aligned} &= \lim_{t \rightarrow \infty} \frac{\mathbf{I}[X_j(0:t); Y_j(0:t), X_{[j+1:n]}(0:t)]}{t+1} \\ &\equiv \mathbf{I}_{\infty}[X_j; Y_j, X_{[j+1:n]}] \geq \mathbf{I}_{\infty}[X_j; Y_j] \\ &\stackrel{(3)}{\geq} d_j, \quad \forall j \in [1 : n], \end{aligned} \tag{19}$$

yielding (6), where (18) follows from the mutual independence of  $X_1, \dots, X_n$ . (If  $X_j$  is continuous-valued then  $\mathbf{h}$  replaces  $\mathbf{H}$ .)

Finally, if  $X_j$  is continuous-valued then  $H[X_j(0:t)] = \infty$ ,  $\forall t \in \mathbb{W}$ , so  $H_\infty[X_j] = \infty$  and (7) automatically holds. On the other hand, if  $X_j$  is discrete-valued then

$$\begin{aligned}
f_{\text{OUT}(\sigma_j),j} &\stackrel{(16)}{=} I_\infty[X_j; Y_j | X_{[j+1:n]}] \\
&\equiv \lim_{t \rightarrow \infty} \left\{ \frac{H[X_j(0:t) | X_{[j+1:n]}(0:t)]}{t+1} \right. \\
&\quad \left. - \frac{H[X_j(0:t) | Y_j(0:t), X_{[j+1:n]}(0:t)]}{t+1} \right\} \\
&\leq \lim_{t \rightarrow \infty} \frac{H[X_j(0:t)] - H[X_j(0:t) | Y_j(0:t), X_{[j+1:n]}(0:t)]}{t+1} \\
&\leq \lim_{t \rightarrow \infty} \frac{H[X_j(0:t)]}{t+1} \equiv H_\infty[X_j],
\end{aligned} \tag{20}$$

where the bound in (20) is due to the nonnegativity of discrete entropy.  $\square$

## APPENDIX C

### NECESSITY PROOF FOR THEOREM 1

Let the arc-signal vector  $S$  be a solution (Def. 2) to the  $n$ -pairs information network problem  $(\Sigma, X, c, d)$ . An informationally feasible  $[1:n]$ -flow (Def. 9)  $f^n$  will be constructed, using upward induction. By Lemma 1,  $f^n$  will then be an  $(X, c, d)$ -feasible multicommodity flow, as desired.

Let  $\Sigma$  be triangularizable (Def. 7) and suppose that  $f^{i-1} = (f_{\alpha,j})_{\alpha \in \mathbf{A}, j \in [1:i-1]} \in \mathbb{R}_{\geq 0}^{|\mathbf{A}|(i-1)}$  is an informationally feasible, acyclic  $[1:i-1]$ -flow for some  $i \in [1:n]$ , noting that the  $\emptyset$ -flow  $f^0$  is informationally feasible. An  $i$ -flow  $(f_{\alpha,i})_{\alpha \in \mathbf{A}} \in \mathbb{R}_{\geq 0}^{|\mathbf{A}|}$  will be constructed in such a way that  $f^i \in \mathbb{R}_{\geq 0}^{|\mathbf{A}|i}$  will be an informationally feasible, acyclic  $[1:i]$ -flow.

On any arc  $\alpha \in \mathbf{A}$ , let

$$r_\alpha := \begin{cases} c_\alpha - \sum_{j=1}^{i-1} f_{\alpha,j} & \text{if } \alpha \in \mathbf{A}_f \\ \infty & \text{if } \alpha \in \mathbf{A}_\infty \equiv \mathbf{A} \setminus \mathbf{A}_f \end{cases} \tag{21}$$

be the residual capacity after subtracting the relevant components of  $f^{i-1}$ . Note that  $r_\alpha \stackrel{(14)}{\geq} 0$  since  $f^{i-1}$  is an informationally feasible  $[1:i-1]$ -flow. The next step is to find an acyclic  $i$ -flow (Def. 8)  $q \in \mathbb{R}_{\geq 0}^{|\mathbf{A}|}$  from  $\sigma_i \rightsquigarrow \tau_i$  that is *a)*  $\leq$  the residual capacity on each arc, and *b)*  $\geq I_\infty[X_i; Y_i | X_{[i+1:n]}]$  on the arc entering  $\tau_i$ . There are two mutually exclusive cases to consider.

#### A. 1st Case: $\exists$ an $i$ -Path with No Finite-Capacity Arcs

Denote this  $i$ -path by  $\pi_e$ , noting that  $r_\alpha = \infty$ ,  $\forall \alpha \in \pi_e$  by the 2nd line of (21). Set the  $i$ -path flows as

$$u_k = \begin{cases} I_\infty[X_i; Y_i | X_{[i+1:n]}] & \text{if } k = e \\ 0 & \text{otherwise} \end{cases}, \quad \forall k \in [1:p], \tag{22}$$

and the cycle flows equal to zero in the decomposition (12) (dropping the  $i$ -subscripts), so that

$$q_\alpha \stackrel{(12)}{=} \sum_{1 \leq k \leq p: \pi_k \ni \alpha} u_k, \quad \forall \alpha \in \mathbf{A}. \tag{23}$$

Evidently  $q$  is acyclic and meets the residual capacity constraint on all arcs in  $\mathbf{A}$ . Furthermore, since every  $i$ -path passes through the single arc entering  $\tau_i$ ,

$$q_{\text{IN}(\tau_i)} \stackrel{(23)}{=} \sum_{1 \leq k \leq p} u_k \stackrel{(22)}{=} u_e = I_\infty[X_i; Y_i | X_{[i+1:n]}], \tag{24}$$

satisfying the conditional information constraint.

*B. 2nd Case: Every  $i$ -Path Has One or More Finite-Capacity Arcs*

Observe first that for any arc-set  $\mathbf{B} \subseteq \mathbf{A}_f$ ,

$$\begin{aligned} \sum_{\beta \in \mathbf{B}} c_\beta &\stackrel{(4)}{\geq} \sum_{\beta \in \mathbf{B}} I^\infty[S_\beta; X] \\ &\equiv \sum_{\beta \in \mathbf{B}} \overline{\lim}_{t \rightarrow \infty} \frac{H[S_\beta(0:t)] - H[S_\beta(0:t)|X(0:t)]}{t+1} \\ &= \sum_{\beta \in \mathbf{B}} \overline{\lim}_{t \rightarrow \infty} \frac{H[S_\beta(0:t)]}{t+1} \end{aligned} \quad (25)$$

$$\begin{aligned} &\geq \overline{\lim}_{t \rightarrow \infty} \frac{1}{t+1} \sum_{\beta \in \mathbf{B}} H[S_\beta(0:t)] \\ &\geq \overline{\lim}_{t \rightarrow \infty} \frac{H[S_{\mathbf{B}}(0:t)]}{t+1} \end{aligned} \quad (26)$$

$$\geq \overline{\lim}_{t \rightarrow \infty} \frac{H[S_{\mathbf{B}}(0:t)|X_{[i+1:n]}(0:t)]}{t+1} \quad (27)$$

$$= \overline{\lim}_{t \rightarrow \infty} \frac{H[S_{\mathbf{B}}(0:t)|X_{[i+1:n]}(0:t)] - H[S_{\mathbf{B}}(0:t)|X(0:t)]}{t+1} \quad (28)$$

$$\begin{aligned} &= \overline{\lim}_{t \rightarrow \infty} \left( \frac{H[S_{\mathbf{B}}(0:t)|X_{[i+1:n]}(0:t)] - H[S_{\mathbf{B}}(0:t)|X_{[i:n]}(0:t)]}{t+1} \right. \\ &\quad \left. + \frac{H[S_{\mathbf{B}}(0:t)|X_{[i:n]}(0:t)] - H[S_{\mathbf{B}}(0:t)|X(0:t)]}{t+1} \right) \\ &= \overline{\lim}_{t \rightarrow \infty} \left( \frac{I[S_{\mathbf{B}}(0:t); X_i(0:t)|X_{[i+1:n]}(0:t)]}{t+1} \right. \\ &\quad \left. + \frac{I[S_{\mathbf{B}}(0:t); X_{[1:i-1]}(0:t)|X_{[i:n]}(0:t)]}{t+1} \right) \\ &\geq I_\infty[X_i; S_{\mathbf{B}}|X_{[i+1:n]}] + I_\infty[X_{[1:i-1]}; S_{\mathbf{B}}|X_{[i:n]}]. \end{aligned} \quad (29)$$

In (25) and (28), the conditional discrete entropies are 0 since, by Lemma 2,  $S_{\mathbf{B}}(0:t)$  is a function of  $X(0:t)$ ; the inequality (26) is due to the subadditivity of joint entropy; and (27) holds because conditioning cannot increase entropy.

Now, consider the residual capacitated digraph  $(\mathbf{V}^i, \mathbf{A}^i, r_{\mathbf{A}^i})$  formed by the  $i$ -bundle.<sup>5</sup> Let  $q$  be an acyclic maximal flow on it under the constraints

$$0 \leq q_\alpha \leq r_\alpha, \quad \forall \alpha \in \mathbf{A}^i. \quad (30)$$

By the *Min-Cut Max-Flow Theorem* (see e.g. [11], Thm. 3.5.3)  $\exists$  an  $i$ -cut  $\mathbf{U} \subset \mathbf{V}^i$ , consisting of every vertex  $\mathbf{v} \in \mathbf{V}^i$  for which  $\exists$  an undirected path  $\pi$  in  $(\mathbf{V}^i, \mathbf{A}^i)$  from  $\sigma_i$  to  $\mathbf{v}$  s.t.

- (*Forward Slack*) every forward-oriented arc  $\alpha$  in  $\pi$  (i.e. pointing from  $\sigma_i$  to  $\mathbf{v}$ ) has  $q_\alpha < r_\alpha$ , and
- (*Backward Flow*) every backward-oriented arc  $\alpha$  in  $\pi$  (pointing from  $\mathbf{v}$  to  $\sigma_i$ ) has  $q_\alpha > 0$ .

As a consequence of this,

$$q_\alpha = r_\alpha, \quad \forall \alpha \in \mathbf{O}^i := \text{OUT}(\mathbf{U}) \cap \mathbf{A}^i, \quad (31)$$

$$q_\alpha = 0, \quad \forall \alpha \in \mathbf{I}^i := \text{IN}(\mathbf{U}) \cap \mathbf{A}^i. \quad (32)$$

<sup>5</sup>Here, arcs are permitted to have  $r_\alpha = 0$ .

Note also that since the cyclic flow components  $w_1, \dots, w_j$  in (12) are zero,

$$q_\alpha = \sum_{1 \leq k \leq p: \pi_k \ni \alpha} u_k, \quad \forall \alpha \in \mathbf{A}^i. \quad (33)$$

The  $i$ -cut  $\mathbf{U}$  evidently depends on the residual capacity vector  $r$ . However, the following *purely structural* statements may be made about it:

- 1) Every arc in  $\mathbf{O}^i$  lies in  $\mathbf{A}_f$ , i.e. is finite-capacity. Otherwise  $q_\alpha \stackrel{(31)}{=} r_\alpha \stackrel{(21)}{=} \infty$ , implying by (33) that  $u_k = \infty$  on some  $i$ -path  $\pi_k$ , which is impossible since every  $i$ -path in this case travels over at least one finite-capacity arc.
- 2) Every arc  $\alpha \in \mathbf{O}^i$  is in an  $i$ -path that exits  $\mathbf{U}$  without re-entering, or else  $\alpha$  is in the  $[1 : i-1]$ -bundle. To see this, suppose that every  $i$ -path  $\pi_k$  passing through  $\alpha$  re-enters  $\mathbf{U}$ . Evidently, it must then pass through some arc  $\beta \in \mathbf{I}^i$ . By (32)  $q_\beta = 0$ , implying by virtue of (33) and nonnegativity that  $u_k = 0$ . From (31) and (33), this implies that  $r_\alpha = 0$ . As  $c_\alpha > 0$ , it must then hold that  $f_{\alpha,j} > 0$  for some  $j \in [1 : i-1]$ . As the  $j$ -flow  $(f_{\alpha,j})_{\alpha \in \mathbf{A}}$  is acyclic by construction,  $\alpha$  must then lie on a  $j$ -path, by (12).
- 3) There must be an  $i$ -path that leaves  $\mathbf{U}$  without re-entering. To see this, suppose in contradiction that every  $i$ -path re-enters  $\mathbf{U}$ . By the preceding argument, all  $i$ -paths must then have associated acyclic flow components  $u_k = 0$ . Pick any  $i$ -path and let  $v$  be the last vertex in  $\mathbf{U}$  that it traverses before leaving  $\mathbf{U}$  without further re-entry. Let  $\omega$  denote its subpath from  $v \rightsquigarrow \tau_i$ . By the definition of  $\mathbf{U}$ , there is an undirected path  $\pi$  from  $\sigma_i$  to  $v$  such that all forward-oriented arcs in it are slack and all backward-oriented arcs carry strictly positive  $q$ -flow. Note also that all vertices before  $v$  in  $\pi$  must also lie in  $\mathbf{U}$ , by construction. From (33), any backward arc in  $\pi$  would have to carry an  $i$ -path flow component  $u_k > 0$ , which would be a contradiction. Consequently, all the arcs in  $\pi$  must be forward-oriented, i.e.  $\pi$  is a directed path in  $\mathbf{U}$  from  $\sigma_i \rightsquigarrow v$ . The concatenation of  $\pi$  with  $\omega$  then yields an  $i$ -path that leaves  $\mathbf{U}$  exactly once, a contradiction.
- 4) Finally, by construction of  $\mathbf{U}$ , every vertex  $v$  in it must lie on an undirected path  $\pi$  from  $\sigma_i$  to  $v$  such that
  - a) every vertex before  $v$  in  $\pi$  is also in  $\mathbf{U}$  (since the subpath from  $\sigma_i$  to  $v$  automatically satisfies the defining forward-slack and backward-flow properties), and
  - b) every reverse-oriented arc in  $\pi$  lies on an  $i$ -path that does not re-enter  $\mathbf{U}$  (since such arcs must by definition carry positive  $q$ -flow, and  $i$ -paths that re-enter  $\mathbf{U}$  carry zero  $q$ -flow).

In other words,  $\mathbf{U}$  is a *viable  $i$ -cut* (Def. 5). By triangularizability (Def. 7),  $\text{OUT}(\mathbf{U})$  contains an arc-set  $\mathbf{B}$  that is a  $[1 : i-1]$ -disjoint, triangular  $[1 : i]$ -choke. Using  $i$ -flow conservation,

$$\begin{aligned} q_{\text{IN}(\tau_i)} &= \sum_{\beta \in \text{OUT}(\mathbf{U})} q_\beta - \sum_{\alpha \in \text{IN}(\mathbf{U})} q_\alpha \stackrel{(31),(32)}{=} \sum_{\beta \in \text{OUT}(\mathbf{U})} r_\beta \\ &\geq \sum_{\beta \in \mathbf{B}} r_\beta \stackrel{(21)}{=} \sum_{\beta \in \mathbf{B}} \left( c_\beta - \sum_{j=1}^{i-1} f_{\beta,j} \right) \\ &\stackrel{(29)}{\geq} \text{I}_\infty [X_i; \mathbf{S}_\mathbf{B} | X_{[i+1:n]}] + \text{I}_\infty [X_{[1:i-1]}; \mathbf{S}_\mathbf{B} | X_{[i:n]}] - \sum_{\beta \in \mathbf{B}, j \in [1:i-1]} f_{j,\beta} \\ &\geq \text{I}_\infty [X_i; \mathbf{S}_\mathbf{B} | X_{[i+1:n]}], \end{aligned} \quad (34)$$

since the informationally feasible  $[1 : i-1]$  flow  $f^{i-1}$  satisfies (15) on  $\mathbf{B}$ , which is automatically a  $[1 : i-1]$ -disjoint, triangular  $[1 : i-1]$ -choke. As  $\mathbf{B}$  is also automatically a downward-whole  $i$ -choke, applying Lemma 4 to the RHS of (34) yields

$$q_{\text{IN}(\tau_i)} \geq \text{I}_\infty [X_i; \mathbf{S}_\mathbf{B} | X_{[i+1:n]}] \geq \text{I}_\infty [X_i; Y_i | X_{[i+1:n]}], \quad (35)$$

as desired.

### C. Construction of $f^i$ in Both Cases

For both cases above, let

$$f_{\alpha,i} := \underbrace{\frac{I_{\infty}[X_i; Y_i | X_{[i+1:n]}]}{q_{\text{IN}(\tau_i)}}}_{=:v} q_{\alpha} \equiv v q_{\alpha}, \quad \forall \alpha \in \mathbf{A}, \quad (36)$$

where  $v \in (0, 1]$ . Clearly,  $f_{\mathbf{A},i}$  is still an acyclic  $i$ -flow since it just a scaled version of  $q$ . Furthermore,

$$\sum_{j=1}^i f_{\alpha,j} \stackrel{(36)}{=} v q_{\alpha} + \sum_{j=1}^{i-1} f_{\alpha,j} \stackrel{(24),(35)}{\leq} q_{\alpha} + \sum_{j=1}^{i-1} f_{\alpha,j} \leq c_{\alpha}.$$

It is next verified that  $f^i = f_{\mathbf{A} \times [1:i]}$  satisfies the remaining conditions (15)–(16) for an informationally feasible  $[1 : i]$ -flow. Let  $\mathbf{E}$  be any  $[1 : i]$ -disjoint, triangular  $[1 : i]$ -choke in  $\Sigma$ . Then

$$\begin{aligned} \sum_{\eta \in \mathbf{E}, j \in [1:i]} f_{\eta,j} &= \sum_{\eta \in \mathbf{E}} f_{\eta,i} + \sum_{\eta \in \mathbf{E}, j \in [1:i-1]} f_{\eta,j} \\ &\stackrel{(36)}{=} \sum_{\eta \in \mathbf{E}} v q_{\eta} + \sum_{\eta \in \mathbf{E}, j \in [1:i-1]} f_{\eta,j} \\ &\stackrel{(33),(23)}{=} v \sum_{\eta \in \mathbf{E}} \left( \sum_{1 \leq k \leq p: \pi_k \ni \eta} u_k \right) + \sum_{\eta \in \mathbf{E}, j \in [1:i-1]} f_{\eta,j}. \end{aligned} \quad (37)$$

The  $[1 : i]$ -disjointness of  $\mathbf{E}$  implies that each  $i$ -path  $\pi_p$  transits over at most one arc in  $\mathbf{E}$ . Combined with the nonnegativity of the  $i$ -path flows  $u_1, \dots, u_p$ , this implies that

$$\begin{aligned} \sum_{\eta \in \mathbf{E}} \left( \sum_{1 \leq k \leq p: \pi_k \ni \eta} u_k \right) &= \sum_{1 \leq k \leq p} u_k \left( \sum_{\eta \in \mathbf{E}: \eta \in \pi_k} 1 \right) \\ &\leq \sum_{1 \leq k \leq p} u_k \equiv q_{\text{IN}(\tau_i)}. \end{aligned} \quad (38)$$

Furthermore,  $\mathbf{E}$  is automatically also a  $[1 : i-1]$ -disjoint, triangular  $[1 : i-1]$ -choke, so by the informational feasibility of  $f^{i-1}$ ,

$$\sum_{\eta \in \mathbf{E}, j \in [1:i-1]} f_{\eta,j} \leq I_{\infty}[X_{[1:i-1]}; \mathbf{S}_{\mathbf{E}} | X_{[i:n]}].$$

Substituting this and (38) into (37) then yields

$$\begin{aligned} \sum_{\eta \in \mathbf{E}, j \in [1:i]} f_{\eta,j} &\leq v q_{\text{IN}(\tau_i)} + I_{\infty}[X_{[1:i-1]}; \mathbf{S}_{\mathbf{E}} | X_{[i:n]}] \\ &\stackrel{(36)}{=} I_{\infty}[X_i; Y_i | X_{[i+1:n]}] + I_{\infty}[X_{[1:i-1]}; \mathbf{S}_{\mathbf{E}} | X_{[i:n]}] \\ &\leq I_{\infty}[X_i; \mathbf{S}_{\mathbf{E}} | X_{[i+1:n]}] + I_{\infty}[X_{[1:i-1]}; \mathbf{S}_{\mathbf{E}} | X_{[i:n]}] \end{aligned} \quad (39)$$

$$\leq I_{\infty}[X_{[1:i]}; \mathbf{S}_{\mathbf{E}} | X_{[i+1:n]}], \quad (40)$$

where (39) is due Lemma 4, since  $\mathbf{E}$  is automatically a downward-whole  $i$ -choke. This confirms that  $f^i$  satisfies (15). As  $f^{i-1}$  is an informationally feasible  $[1 : i-1]$  flow, (16) is satisfied  $\forall j \in [1 : i-1]$ . By flow conservation,

$$\begin{aligned} f_{\text{OUT}(\sigma_i),i} &= f_{\text{IN}(\tau_i),i} \\ &\stackrel{(36)}{=} v q_{\text{IN}(\tau_i)} = I_{\infty}[X_i; Y_i | X_{[i+1:n]}], \end{aligned}$$

verifying (16) when  $j = i$ . Thus  $f^i$  is an informationally feasible  $[1 : i]$ -flow.

By upward induction,  $f^n$  is an informationally feasible  $[1 : n]$ -flow. If the reverse structure  $\Sigma'$  is triangularizable, the induction goes through identically, but with  $(\mathbf{A}'_f, \mathbf{A}'_\infty, \mathbf{P}')$  replacing  $(\mathbf{A}_f, \mathbf{A}_\infty, \mathbf{P})$ . The informationally feasible  $[1 : n]$ -flow  $f'^n$  obtained on  $\Sigma'$  then yields an informationally feasible  $[1 : n]$ -flow  $f^n$  on  $\Sigma$ , using  $f_{(\mu, \nu), j} = f'_{(\nu, \mu), j}$ ,  $\forall (\mu, \nu) \in \mathbf{A}$  and  $j \in [1 : n]$ .

By Lemma 1,  $f^n$  is automatically an  $(X, c, d)$ -feasible multicommodity flow, as desired.

#### APPENDIX D SUFFICIENCY PROOF FOR THEOREM 1

In this section, the sufficiency part of Thm. 1 is established. For reasons of space, only a sketch is provided.

Suppose  $f$  is an  $(X, c, d)$ -feasible multicommodity flow (5)–(8) on  $\Sigma$ . In the decomposition (12) for each  $i$ -flow  $f_{\mathbf{A}, i}$ , no cycle flow can enter any sink, since it has no departing arcs. Consequently, the cycle flows may be taken to be zero in (12) without violating (5)–(8), yielding

$$f_{\alpha, i} = \sum_{1 \leq k \leq p_i : \pi_{k, i} \ni \alpha} u_{k, i}, \quad (41)$$

where  $\pi_{1, i}, \dots, \pi_{p_i, i}$  are the  $i$ -paths and  $u_{1, i}, \dots, u_{p_i, i} \geq 0$ , the  $i$ -path flows.

As  $\sum_{k=1}^{p_i} u_{k, i} \stackrel{(6)}{\leq} H_\infty[X_i]$ , Slepian-Wolf-style coding ideas can be used, at the vertex succeeding each source  $\sigma_i$ , to causally produce  $p_i$  mutually independent data-streams  $Z_{i, 1}, \dots, Z_{i, p_i}$  from each source signal  $X_i$ , with the  $k$ -th data-stream designed to have average bit-rate  $H^\infty[Z_{i, k}] = H_\infty[Z_{i, k}] = u_{k, i}$  and routed along the  $i$ -path  $\pi_{k, i}$ . On every arc  $\alpha \in \mathbf{A}$  apart from the ones leaving sources, the arc-signal is then represented by a vector  $S_\alpha = (Z_{i, k})_{i \in [1 : n], k \in [1 : p_i] : \pi_{k, i} \ni \alpha}$  with mutually independent components.<sup>6</sup> The arc-signals leaving sources are set to the respective source signals to satisfy (2). It may be verified that  $S$  is setwise causal (Def. 1), since every signal  $Z_{i, k}$  is routed along an acyclic path. In addition,  $\forall \alpha \in \mathbf{A}_f$ ,

$$I^\infty[S_\alpha; X] = H^\infty[S_\alpha] \equiv H^\infty \left[ (Z_{i, k})_{i \in [1 : n], k \in [1 : p_i] : \pi_{k, i} \ni \alpha} \right] \quad (42)$$

$$\begin{aligned} &\leq \sum_{i \in [1 : n], k \in [1 : p_i] : \pi_{k, i} \ni \alpha} H^\infty[Z_{i, k}] \\ &= \sum_{i \in [1 : n], k \in [1 : p_i] : \pi_{k, i} \ni \alpha} u_{k, i} \\ &\stackrel{(41)}{=} \sum_{i \in [1 : n]} f_{\alpha, i} \stackrel{(5)}{\leq} c_\alpha, \end{aligned} \quad (43)$$

where the 1st equality in (42) holds because  $S_\alpha$  is causally determined by  $X$  by Lemma 2 and (43) is due to the subadditivity of entropy. Furthermore,

$$\begin{aligned} I_\infty[X_i; Y_i] &\equiv I_\infty[X_i; S_{\text{IN}(\tau_i)}] = I_\infty[X_i; Z_{i, 1}, \dots, Z_{i, p_i}] \\ &= H_\infty[Z_{i, 1}, \dots, Z_{i, p_i}] = \sum_{k=1}^{p_i} H_\infty[Z_{i, k}] \\ &= \sum_{k=1}^{p_i} u_{k, i} \stackrel{(41)}{=} f_{\text{IN}(\tau_i), i} \stackrel{(6)}{\geq} d_i, \end{aligned} \quad (44)$$

where (44) is due to the fact that  $Z_{i, 1}, \dots, Z_{i, p_i}$  are causally determined by  $X_i$ , and are also mutually independent. Consequently,  $S$  is a solution to the  $n$ -pair information network problem  $(\Sigma, X, c, d)$ , establishing achievability (Def. 2).

<sup>6</sup>If an arc is not on any  $i$ -path, then its arc-signal may be taken to be a constant.